RESEARCH NOTES

Formula and 2-adic valuation of $L(1)$ of elliptic curves with CM by $\sqrt{-3}$*

QIU Derong

(Center for Advanced Study, Tsinghua University, Beijing 100084, China)

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Abstract For the rational integers $\lambda\equiv 1$, 3, or 5 (mod 6), considering elliptic curves $y^2=x^3-2^4 3^3 D^\lambda$ over the field $\mathbb{Q}(\sqrt{-3})$, the formula for the value at $s=1$ of Hecke L-series attached to such elliptic curves, expressed as a finite sum of values of Weierstrass $\wp$-functions, is obtained. Moreover, when $\lambda\equiv 3$ (mod 6), the lower bounds of 2-adic valuations of these values are also obtained. These results are consistent with the predictions of the conjecture of Birch and Swinnerton-Dyer in a sense, and have generalized and advanced some results in recent literature.

Keywords: elliptic curve, L-series, complex multiplication, Birch and Swinnerton-Dyer conjecture.

Consider elliptic curves

$$E_{D^\lambda}:\ y^2 = x^3 - 2^4 3^3 D^\lambda \qquad (1)$$

over the quadratic imaginary field $\mathbb{Q}(\sqrt{-3})$ with rational integer $\lambda\not\equiv 0$ (mod 6). Recently, for the non-trivial even λ cases, i.e. for the cases of $\lambda\equiv 2$ or 4 (mod 6), the values at $s=1$ of Hecke L-series attached to E_{D^λ} were studied systematically[1~3]. The formulae (expressed via the Weierstrass $\wp$-function) for these values were given, and the uniform lower bounds for 3-adic valuations of the values were also obtained.

In this paper, we study the elliptic curves E_{D^λ} in (1) for the odd λ cases, i.e. the cases of $\lambda\equiv 1$, 3, or 5 (mod 6), and give a uniform formula for the special values at $s=1$ of Hecke L-series attached to such elliptic curves, expressed as a finite sum of values of Weierstrass $\wp$-functions. In particular, when $\lambda\equiv 3$ (mod 6), we obtain the lower bounds of the 2-adic valuations of the values at $s=1$ of Hecke L-series attached to E_{D^λ}.

Our results advance the results in Refs. [1~4] about E_{D^λ} for the even λ, and in Refs. [1,2,5,6] for elliptic curves $y^2=x^3-D_1^\gamma x$ ($\gamma\in\mathbb{Z}$) over Gaussian field $\mathbb{Q}(\sqrt{-1})$. In particular, our results are consistent with the predictions of B-SD conjecture in a certain sense (see Remark 1).

1 Formulae of $L(1)$ attached to E_{D^λ}

Throughout, let $K=\mathbb{Q}(\sqrt{-3})$, $\tau=(-1+\sqrt{-3})/2$ be a primitive cubic root of unity, and $O_K=\mathbb{Z}[\tau]$ the ring of integers of K. We study the elliptic curves

$$E_{D^\lambda}:\ y^2 = x^3 - 2^4 3^3 D^\lambda \text{ with } D=\pi_1\cdots\pi_n, \qquad (2)$$

where $\pi_k\equiv 1$ (mod 12) are distinct prime elements of $O_K(k=1,\cdots,n)$, $\lambda\in\mathbb{Z}$ and $\lambda\equiv 1$, 3, or 5 (mod 6). Let $S=\{\pi_1,\cdots,\pi_n\}$. For any subset T of the set $\{1,\cdots,n\}$, define

$$D_T=\prod_{k\in T}\pi_k,\quad \hat{D}_T=D/D_T,$$

and put $D_\varnothing=1$ when $T=\varnothing$ (empty set). (3)

Let $\psi_{D_T^\lambda}$ be the Hecke character (i.e. Grössencharacter) of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D_T^\lambda}$: $y^2=x^3-2^4 3^3 D_T^\lambda$, and let $L_S(\bar{\psi}_{D_T^\lambda},s)$ be the Hecke L-series of $\bar{\psi}_{D_T^\lambda}$ (the complex conjugate of $\psi_{D_T^\lambda}$) with the Euler factors omitted at all primes in S. For the definition of such Hecke L-series attached to CM elliptic curves, see Ref. [7]. We have the following uniform formulae for special values $L_S(\bar{\psi}_{D_T^\lambda},1)$ of the

* Supported by China Postdoctoral Science Foundation (Grant No. 65 (2001))
E-mail: derong@castu.tsinghua.edu.cn

above Hecke L-series at $s=1$ which are expressed by the values of Weierstrass $\wp$-functions.

Theorem 1. For any factor D_T of $D=\pi_1\cdots\pi_n\in\mathbb{Z}[\tau]$ and any rational integer $\lambda\equiv 1$, 3, or 5 (mod 6) as above, let $\psi_{D_T^\lambda}$ be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D_T^\lambda}$: $y^2=x^3-2^4 3^3 D_T^\lambda$. Then we have

$$\frac{D}{\omega_0}\left(\frac{12}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1) = \frac{1}{2\sqrt{3}}\sum_{c\in C}\left(\frac{c}{D_T}\right)_6^\lambda \frac{1}{\wp\left(\frac{4c\omega_0}{D}\right)-1} + \frac{1}{3\sqrt{3}}\sum_{c\in C}\left(\frac{c}{D_T}\right)_6^\lambda,$$

where $(-)_6$ is the sextic residue symbol, C any complete set of representatives of the relatively prime residue classes of O_K modulo D, and $\wp(z)$ the Weierstrass $\wp$-function satisfying $\wp'(z)^2=4\wp(z)^3-1$ with period lattice $L_{\omega_0}=\omega_0 O_K$ (corresponding to the elliptic curve $y^2=x^3-1/4$), and $\omega_0=3.059908\cdots$ is an absolute constant.

Now for general rational integers λ with $\lambda\not\equiv 0$ (mod 6), we define a function $\sigma(\lambda)=\frac{1-(-1)^\lambda}{2}$. Then together with the results in Refs. [1~3] about E_{D^λ}: $y^2=x^3-2^4 3^3 D^\lambda$ for the cases of $\lambda\equiv 2$, or 4 (mod 6), we obtain the following results for general λ (when $\lambda\equiv 0$ (mod 6), E_{D^λ} is $\mathbb{Q}(\sqrt{-3})$-isomorphic to E_1: $y^2=x^3-2^4 3^3$, which is the trivial case).

Theorem 2. Let λ be any rational integer and $\lambda\not\equiv 0$ (mod 6). $D=\pi_1\cdots\pi_n$, where $\pi_k\equiv 1$ (mod $6\cdot 2^{\sigma(\lambda)}$) are distinct prime elements of $\mathbb{Z}[\tau]$ $(k=1,\cdots,n)$. For any factor D_T of D as defined in (3), let $\psi_{D_T^\lambda}$ be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D_T^\lambda}$: $y^2=x^3-2^4 3^3 D_T^\lambda$. Then we have

$$\frac{D}{\omega_0}\left(\frac{3\cdot 4^{\sigma(\lambda)}}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1) = \frac{1}{2\sqrt{3}}\sum_{c\in C}\left(\frac{c}{D_T}\right)_6^\lambda \frac{1}{\wp\left(\frac{4^{\sigma(\lambda)}c\omega_0}{D}\right)-1} + \frac{1}{3\sqrt{3}}\sum_{c\in C}\left(\frac{c}{D_T}\right)_6^\lambda,$$

where $(-)_6$, C, $\wp(z)$ and ω_0 are all the same as in Theorem 1.

2 The 2-adic valuations of $L(1)$ of E_{D^3}

Now we turn to study 2-adic valuations of the special values at $s=1$ of the Hecke L-series attached to E_{D^λ} in (2) for the case of $\lambda\equiv 3$ (mod 6). Up to $\mathbb{Q}(\sqrt{-3})$-isomorphism, we only need to consider elliptic curves

$$E_{D^3}:\ y^2=x^3-2^4 3^3 D^3 \text{ with } D=\pi_1\cdots\pi_n, \quad (4)$$

where $\pi_k\equiv 1$ (mod 12) are distinct prime elements of $\mathbb{Z}[\tau]$ $(k=1,\cdots,n)$.

Let ψ_{D^3} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve E_{D^3} in (4), and let $L(\bar{\psi}_{D^3},s)$ denote the Hecke L-series of $\bar{\psi}_{D^3}$ (the complex conjugate of ψ_{D^3}). Also we let ω_0 denote the real period of the Weierstrass $\wp$-function in Theorem 1. Then by Ref. [8] (or [9]) it can be easily seen that the values $L(\bar{\psi}_{D^3},1)/\omega_0$ are all algebraic numbers.

Let $\mathbb{Q}_2$ be the completion of $\mathbb{Q}$ at 2-adic valuation, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_2$ the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_2$ respectively; and let v_2 be the normalized 2-adic additive valuation of $\overline{\mathbb{Q}}_2$ (i.e. $v_2(2)=1$). Fix an isomorphic embedding $\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_2$. Then $v_2(\alpha)$ is defined for any algebraic number α in $\overline{\mathbb{Q}}$. The value $v_2(\alpha)$ for $\alpha\in\overline{\mathbb{Q}}$ depends on the choice of the embedding $\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_2$, but this does not affect our discussion in the following.

Theorem 3. Let $D=\pi_1\cdots\pi_n$, where $\pi_k\equiv 1$ (mod 12) are distinct prime elements of $\mathbb{Z}[\tau]$ $(k=1,\cdots,n)$, and let ψ_{D^3} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve E_{D^3}: $y^2=x^3-2^4 3^3 D^3$. Then for the 2-adic valuation of $L(\bar{\psi}_{D^3},1)/\omega_0$ we have

$$v_2(L(\bar{\psi}_{D^3},1)/\omega_0)\geqslant n.$$

Remark 1.

(i) Our results in Theorem 3 are consistent with the predictions of B-SD conjecture in a certain sense. In fact, by the methods in Ref. [10] (or [7]), we can prove that under our hypothesis of D the Tamagawa factor $c_v=1$, 2, or 4 for any finite place v satisfying $v\mid N_E$ and $v\nmid 6$, where N_E is the conductor of $E=E_{D^3}$ in Theorem 3. Let $\Omega \in \mathbb{C}^{\times}$ be a O_K-generator of the period lattice of a minimal model of E. Consider the case $L(E/K, 1) \neq 0$ (The case $L(E/K, 1)=0$ does not need to be considered since $v_2(L(E/K, 1))=\infty$). Then the B-SD conjecture[11,12] predicts that $L(E/K,1)/\Omega\bar{\Omega}$ has the factor

$$\prod_{v \mid N(E) \text{ and } v \nmid 6} c_v = 2^m \quad \text{for certain exponent } m.$$

Further, when the number $n(D)$ of distinct prime factors of D becomes greater, $N(E)$ would have more prime factors v, so m would become greater, this is consistent with our results of Theorem 3 since we have[7]

$$L(E/K,1)=L(\psi,1)L(\bar{\psi},1), \quad \text{where } \psi=\psi_{D^3}.$$

(ii) For the elliptic curves E_{D^λ} with $\lambda=1$, or 5 in (2), we could also prove by the same methods in Ref. [10] (or [7]) that the Tamagawa factor $c_v=1$ for any finite place $v \mid N_E$ and $v \nmid 6$, where N_E is the conductor of $E=E_{D^\lambda}$. This means that the product of the Tamagawa factors

$$\prod_{v \mid N(E) \text{ and } v \nmid 6} c_v$$

does not increase. So we do not consider the problem of $p(=2$, or $3)$-adic valuation for these cases.

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